Solution to Assignment 2

Section 6.2

5. Let $f(x) := x^{1/n} - (x-1)^{1/n}$, for $x \geq 1$.
Then $f'(x) = \frac{1}{n}x^{1/n-1} - \frac{1}{n}(x-1)^{1/n-1}$ for $x > 1$.
Define $g(t) := t^{1/n-1}$ for $t > 0$, $g'(t) = \left(\frac{1}{n} - 1\right) t^{1/n-2} < 0$ since $n \geq 2$.
Then for $x > 1$, $f'(x) = \frac{1}{n}g(x) - \frac{1}{n}g(x-1) < 0$. Hence f is strictly decreasing for $x > 1$.
Note $a > b > 0$, then $a/b > 1$, hence $f(a/b) < \lim_{x \to 1^+} f(x) = f(1)$, by continuity,
i.e. $\left(\frac{a}{b}\right)^{1/n} - \left(\frac{a}{b} - 1\right)^{1/n} < 1 - (1-1) = 1 \quad \Rightarrow \quad a^{1/n} - b^{1/n} < (a-b)^{1/n}$.

9. For $x \neq 0$, $f(x) = 2x^4 + x^4 \sin\frac{1}{x} \geq 2x^4 - x^4 = x^4 > 0 = f(0)$
Hence f has an absolute minimum at $x = 0$.
For $x \neq 0$, $f'(x) = 8x^3 + 4x^3 \sin\frac{1}{x} + x^4 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) = x^2\left(8x + 4x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$
Define $a_n := 1/2n\pi$ and $b_n := 1/(2n\pi + \pi/2)$ with $\lim a_n = \lim b_n = 0$.
Then $f'(a_n) = \left(\frac{1}{2n\pi}\right)^2\left(\frac{8}{2n\pi} - 1\right) < \left(\frac{1}{2n\pi}\right)^2\left(\frac{8}{6n} - 1\right) < 0$ if $n \geq 2$
$f'(b_n) = \left(\frac{1}{2n\pi + \pi/2}\right)^2\left(\frac{8}{2n\pi + \pi/2} - \frac{4}{2n\pi + \pi/2}\right) > 0 \ \ \forall\ n.$
Let $\varepsilon > 0$. Then $\exists\ N_1, N_2 \in \mathbb{N}$ s.t. $|a_{N_1}| < \varepsilon$ and $|b_{N_2}| < \varepsilon$, i.e. $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)$.
WLOG assume $N_1 \geq 2$. Hence $f'(a_{N_1}) < 0, f'(b_{N_2}) > 0$ with $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)\ \forall\ \varepsilon > 0$.
Hence the derivative has both positive and negative values in every nbd of 0.

10. $\frac{g(x) - g(0)}{x - 0} = \frac{x + 2x^2\sin(1/x)}{x} = 1 + 2x\sin\frac{1}{x} \quad \Rightarrow \quad g'(0) = 1 + 2\lim_{x \to 0} x \sin\frac{1}{x} = 1 + 2(0) = 1.$
For $x \neq 0, g'(x) = 1 + 4x\sin(\frac{1}{x}) - 2\cos(\frac{1}{x})$. Define $a_n := 1/2n\pi$ and $b_n := 1/(2n\pi + \pi/2)$ with $\lim a_n = \lim b_n = 0$.
Then $g'(a_n) = 1 - 2\cos 2n\pi = -1 < 0$, and
$g'(b_n) = 1 + 4(\frac{1}{2n\pi + \frac{\pi}{2}}) > 0.$

Let $\varepsilon > 0$. Then $\exists\ N_1, N_2 \in \mathbb{N}$ s.t. $|a_{N_1}| < \varepsilon$ and $|a_{N_2}| < \varepsilon$, i.e. $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)$.
Hence $g'(a_{N_1}) > 0, g'(b_{N_2}) < 0$ with $a_{N_1}, b_{N_2} \in (-\varepsilon, \varepsilon)\ \forall\ \varepsilon > 0$.
Thus g cannot be monotonic on $(-\varepsilon, \varepsilon)\ \forall\ \varepsilon > 0$, (read Theorem 6.2.7 carefully), i.e. any nbd of 0.

11. Take $f(x) := \sqrt{x}$ is continuous on $[0, 1]$ and hence uniformly continuous on $[0, 1]$.
For $x > 0$, $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded, which can be proved by putting $x = x_n := \frac{1}{4n^2} \to 0$.

12. Assume $\exists$ such function f. Then $f|_{[-1,1]}$ is differentiable on $[-1, 1]$.
By Darboux theorem, $\exists\ c \in (-1, 1)$ s.t. $f'(c) = h(c) = 1/2$, which is contradiction, as h takes only values 0 and 1. Hence such function does not exist.

Consider $f(x) = \begin{cases} x, & x \geq 0 \\ 0, & \text{o.w..} \end{cases}$, $g(x) = \begin{cases} x, & x \geq 0 \\ 1, & \text{o.w..} \end{cases}$

Then $f(x) - g(x) = \begin{cases} 0, & x \geq 0 \\ -1, & \text{o.w..} \end{cases}$ is not a constant but $f'(x) = g'(x) = h(x)$ for $x \neq 0$.

17. By looking at the function $h = g - f$, it is equivalent to showing $h' \geq 0$ and $h(0) = 0$ implies $h(x) \geq 0$. But this follows from the fact that $h' \geq 0$ implies h is increasing. As $h(0) = 0, h$ must be non-negative for all $x \geq 0$.

18. Let $\varepsilon > 0$. Then $\exists\ \delta$ s.t.

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon, \quad \forall\ 0 < |x - c| < \delta.$$

For $x < c < y$ inside $(c - \delta, c + \delta)$,

$$-\varepsilon(y - c) < f(y) - f(c) - f'(c)(y - c) < \varepsilon(y - c)$$

$$-\varepsilon(x - c) > f(x) - f(c) - f'(c)(x - c) > \varepsilon(x - c)$$

$$-\varepsilon(y - x) < f(y) - f(x) - f'(c)(y - x) < \varepsilon(y - x)$$

$$\left| \frac{f(y) - f(x)}{y - x} - f'(c) \right| < \varepsilon.$$

19. Let $\varepsilon > 0$. By uniform differentiability, $\exists\ \delta := \delta(\varepsilon) > 0$ s.t. if $0 < |x - y| < \delta$, then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \frac{\varepsilon}{2}, \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \frac{\varepsilon}{2}$$

$$|f'(x) - f'(y)| \leq \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence f' is continuous on I.

Supplementary Problems

1. Let f be a function defined on $\mathbb{R}$. It is called a periodic function if there is a non-zero number T such that $f(x + T) = f(x)$ for all x. The number T is called a period of f.

   (a) Show that $nT, n \neq 0, \in \mathbb{Z}$, is also a period if f has a period T.

   (b) Let f be differentiable. Show that f must be constant if it has a sequence of periods $\{T_n\}, T_n \to 0$.

   (c) (Optional) Let f be differentiable. Show that if f is non-constant, there exists a positive period L satisfying, if T is another period of f, then $T = nL$ for some integer n. This L is called the minimal period of f.

   Solution. (a) When $n \geq 2$, $f(x + nT) = f(x + (n-1)T + T) = f(x + (n-1)T) = f(x + (n-2)T + T) = f(x + (n-2)T) = \cdots = f(x)$. On the other hand, $f(x - T) = f(x - T + T) = f(x)$, so $-T$ is also a period if T is.

   (b) Let $T_n \to 0$ be periods and x be any point. We have

$$f'(x) = \lim_{n \to \infty} \frac{f(x + T_n) - f(x)}{T_n} = 0\ ,$$

so $f' \equiv 0$ implies that f is a constant.

(c) By (b), the number $T^* = \inf\{T : T \text{ is a positive period}\}$ is positive. For any positive period T, we have $T = nT^* + P$ for some $P \in [0, T^*)$ and $n \geq 1$. It is easy to see that P is a period if it is non-zero. Since T^* is the infimum of all periods, $P = 0$.

Note: In this proof we used the fact that f is differentiable everywhere. In fact, one can show that a periodic function which is non-constant and continuous at one point has a minimal period. On the other hand, the function $g(x) = 1$ when x is rational and $g(x) = 0$ otherwise is a nowhere continuous function. Any positive rational number is a period of this function, so it does not have a minimal period.

2. Let f be a differentiable function defined on $(0, \infty)$. Suppose f satisfies $|f(x)| \leq C\sqrt{x}$ for all $x \in (0, \infty)$ for some constant $C > 0$. Show that there exists a sequence of numbers $\{x_n\}, x_n \to \infty$, such that $f'(x_n) \to 0$ as $n \to \infty$.

   Solution. Applying Mean-Value Theorem to the intervals $[n, 2n]$, we find $x_n \in (n, 2n)$ such that $|f'(x_n)| = |(f(2n) - f(n))|/(2n - n) \leq (\sqrt{2n} - \sqrt{n})/n = 1/(\sqrt{2n} + \sqrt{n}) \to 0$.

3. (a) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Suppose that p has n real roots. Show that p' has $n - 1$ real roots.

   (b) (Optional) What happens when p does not have n real roots? In this case, there are complex roots. Could you make a guess on the roots of p'?

   Solution. (a) Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ be the k distinct real roots of $p(x) = 0, m_i > 0$ be the mulitiplicity of α_i. By Rolle's theorem or Mean value theorem, $\exists \beta_i \in (\alpha_i, \alpha_{i+1})$ such that
   $$p'(\beta_i) = 0, i = 1, 2, \ldots, k - 1.$$
   Note that $\beta_i \neq \beta_j$ if $i \neq j$. If α_i is a real root of multiplicity m_i, then α_i will be a real root of $p'(x)$ having mulitiplicity $m_i - 1$. In total there are $\sum_{i=1}^{k}(m_i - 1) + k - 1 = \sum_{i=1}^{k} m_i - 1 = n - 1$ real roots for $p'(x)$.

   (b) p' may still have $n - 1$ real roots. For example, $p(x) = x^2 + 1$ which has no real roots. $p'(x) = 2x + 1$ has $-1/2$ as a root. However, it may happen that p' does not have $n - 1$ real roots. For instance, $p(x) = (x^2 + 1)^2$. $p'(x) = 4x(x^2 + 1)$ which has only one real root instead of three. A general theorem in complex analysis says a polynomial always has n many complex roots (including multiplicity). The roots of p' are contained inside the convex hull of the roots of p, that is, the smallest convex set in the complex plane containing all roots of p. It reduces to (a) when all roots of p are real. Wiki for Guass-Lucas Theorem. The proof of this theorem is not difficult.

4. It has been shown that a differentiable function f on (a, b) satisfying $f'(x) = 0$ everywhere must be a constant. Show that this result is not true when the assumption is relaxed to the right derivative of f exists and $f'_+(x) = 0$ everywhere.

   Solution. The function $f(x) = -1, x \in (-1, 0)$ and $f(x) = 1, x \in (0, 1)$ satisfies $f'_+(x) = 0$ for all $x \in (-1, 1)$. But it is not a constant.